MATH 245 S24, Exam 3 Solutions

1. Carefully define the following terms: symmetric difference, union.

   Let S, T be sets. Their symmetric difference is the set given by $\{x : (x \in S \wedge x \notin T) \vee (x \in T \wedge x \notin S)\}$. Their union is the set given by $\{x : x \in S \vee x \in T\}$.

2. Carefully define the following terms: disjoint, antisymmetric

   Two sets S, T are disjoint if $S \cap T = \emptyset$. Let S be a set. Relation R on S is antisymmetric if it satisfies the property $\forall x, y \in S,\ (xRy \wedge yRx) \rightarrow x = y$.

3. Find a partition of $\mathbb{N}$ into infinitely many parts, each of a different cardinality (from each other). You need not prove it is a partition, just find it.

   Many solutions are possible; here is one: $\{S_1, S_2, S_3, S_4, \ldots\}$, where $S_1 = \{1\}, S_2 = \{2, 3\}, S_3 = \{4, 5, 6\}, S_4 = \{7, 8, 9, 10\}, \ldots$. That is, $S_n = \{x \in \mathbb{N} : \frac{n(n-1)}{2} < x \leq \frac{n(n+1)}{2}\}$. You need not find a closed form for S_n like this, but if you don't your pattern needs to be very clear. This is a partition because each element of $\mathbb{N}$ is in exactly one such part S_n. Also $|S_n| = n$, so they are all of different cardinalities.

4. Let $R = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 2y\}, S = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 6y\}, T = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}, x = 3y\}$. Prove or disprove that $R \cap S = T$.

   The statement is false, and to disprove we need an explicit counterexample, some $x^\star \in T$ with $x^\star \notin R \cap S$. Many are possible, I choose $x^\star = 3$. $3 \in T$ because $3 = 3 \cdot 1$ and $1 \in \mathbb{Z}$. Now, $3 \notin R$ since $3 = 2y$ only has solution $y = 3/2$ which is not an integer. Since $3 \notin R$, also $3 \notin R \cap S$. [Proof 1: If $3 \in R \cap S$, then $3 \in R \wedge 3 \in S$, and by simplification $3 \in R$, which is a contradiction.] [Proof 2: By addition $3 \notin R \vee 3 \notin S$. By De Morgan's Law $\neg(3 \in R \wedge 3 \in S)$, so $\neg 3 \in R \cap S$, so $3 \notin R \cap S$.]

5. Let S, T be sets, and suppose that $S \setminus T = T \setminus S$. Prove that $S \subseteq T$.

   METHOD 1: Let $x \in S$. We now have two cases, based on whether or not $x \in T$. If $x \in T$, we are happy and done. If instead $x \notin T$, then by conjunction $x \in S \wedge x \notin T$, so $x \in S \setminus T$. Since $S \setminus T = T \setminus S$, in fact $x \in T \setminus S$. Hence $x \in T \wedge x \notin S$, and by simplification $x \in T$. In both cases $x \in T$. ALTERNATE ENDING: Once we get $x \in T$ in the second case, we combine with $x \notin T$ (which holds in the second case), to conclude that the second case never happens.

   METHOD 2: Let $x \in S$. We will prove $x \in T$ by contradiction. So, we suppose $x \notin T$. By conjunction $x \in S \wedge x \notin T$, so $x \in S \setminus T$. Since $S \setminus T = T \setminus S$, in fact $x \in T \setminus S$. Hence $x \in T \wedge x \notin S$, and by simplification $x \notin S$. This is a contradiction, proving $x \in T$.

   METHOD 3 (found by a student): Let $x \in S$. By addition, $x \in S \vee x \notin T$. By double negation, $(\neg x \notin S) \vee (\neg x \in T)$. By De Morgan's Law, $\neg(x \notin S \wedge x \in T)$, and hence $\neg x \in (T \setminus S)$. Because $T \setminus S = S \setminus T$, $\neg x \in (S \setminus T)$. Hence $\neg(x \in S \wedge x \notin T)$. By De Morgan's Law again, $x \notin S \vee x \in T$. By disjunctive syllogism with $x \in S$, $x \in T$.

6. Let A, B be nonempty sets, and suppose that $A \times B \subseteq B \times A$. Prove that $A = B$.
First, let $x \in A$. Since B is nonempty, let $y \in B$. We have $(x, y) \in A \times B$, and since $A \times B = B \times A$, also $(x, y) \in B \times A$. So $x \in B$. This proves $A \subseteq B$.

Second, let $x \in B$. Since A is nonempty, let $z \in A$. We have $(z, x) \in A \times B$, and since $A \times B = B \times A$, also $(z, x) \in B \times A$. So, $x \in A$. This proves $B \subseteq A$.

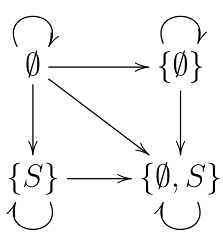
For full credit, a solution must use the hypothesis that A, B are nonempty.

7. Let S, U be sets with $S \subseteq U$. Prove that $S \subseteq (S^c)^c$.
NOTE: This is part of Theorem 9.2. Do not use this theorem to prove itself!
METHOD 1: Let $x \in S$. We apply double negation to get $\neg\neg x \in S$. We apply addition to get $(\neg x \in U) \vee (\neg(\neg x \in S))$. We apply De Morgan's Law for propositions (Thm 2.11) in the reverse direction from usual to get $\neg(x \in U \wedge (\neg x \in S))$. Hence we get $\neg x \in S^c$. We combine $x \in S$ with $S \subseteq U$ to get $x \in U$. By conjunction we get $x \in U \wedge \neg(x \in S^c)$. Finally we get $x \in (S^c)^c$.

METHOD 2 (found by a student): Let $x \in S$. Because $S \subseteq U$ we have $x \in U$. By conjunction $x \in U \wedge x \in S$. By addition $(x \in U \wedge x \in S) \vee (x \in U \wedge x \notin U)$. By distributivity (in reverse) $x \in U \wedge (x \in S \vee x \notin U)$. By double negation $x \in U \wedge (\neg x \notin S \vee \neg x \in U)$. By De Morgan's Law $x \in U \wedge \neg(x \notin S \wedge x \in U)$ and hence $x \in U \wedge \neg(x \in U \setminus S)$. Thus $x \in U \wedge \neg x \in S^c$. Hence $x \in U \setminus S^c$, and so $x \in (S^c)^c$.

For problems 8-10, let $S = \{a\}$, $V = 2^S$, and $W = 2^V$. Define relation R on W via $R = \{(x, y) : x \subseteq y\}$. Each of these problems has two parts.

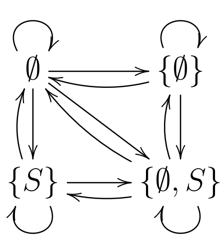
8. Draw the digraph for relation R. Also, determine $|R|$.



Note that $V = \{\{\}, \{a\}\} = \{\emptyset, S\}$ and $W = \{\emptyset, \{\emptyset\}, \{S\}, \{\emptyset, S\}\}$. We have $|R| = 9$, as there are nine directed edges. It's important to keep clear the difference between $\{\} = \emptyset$ and $\{\{\}\} = \{\emptyset\}$!.

9. Prove or disprove that R is reflexive. Also, prove or disprove that R is symmetric.
R is reflexive, because $x \subseteq x$ for *all* sets x, and in particular for all sets $x \in W$. R is not symmetric; to disprove requires an explicit $x, y \in W$ such that $(x, y) \in R$ but $(y, x) \notin R$. Five choices are available, I will pick $x = \{\emptyset\}, y = \{\emptyset, S\}$. Note that $(x, y) \in R$ (there is only one element of x, namely $\emptyset$, and it is an element of y) and $(y, x) \notin R$ (since $S \in y$ and $S \notin x$, so $y \not\subseteq x$).

10. Let R_{symm} denote the symmetric closure of R. Draw the digraph for relation R_{symm}. Also, prove or disprove that R_{symm} is transitive.



R_{symm} has every possible edge, except for the two between $\{\emptyset\}$ and $\{S\}$. It is not transitive, and to prove this requires using one of those missing edges in constructing an explicit counterexample. Here is one possible way: take $x = \{S\}, y = \{\emptyset, S\}, z = \{\emptyset\}$. We have $(x, y) \in R_{symm}$ and $(y, z) \in R_{symm}$ but $(x, z) \notin R_{symm}$.